ON ENGEL-TYPE THEOREMS FOR SEMIGROUPS OF MATRICES

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Abstract. We obtain Engel-type theorems for commutative, nilpotent and quasinilpotent semigroups of matrices with complex or real nonnegative entries both in conventional and tropical linear algebra.

1. Preliminaries

We consider semigroups of matrices with entries in 1) the complex field, 2) the semifield of nonnegative real numbers with usual arithmetics, 3) the semifield of nonnegative numbers $\mathbf{R}_+(\max)$ with idempotent addition $a \oplus b := \max(a, b)$ and the usual multiplication $ab := a \times b$.

The latter example is known as the **max-times semifield**; this semifield is isomorphic to the so-called tropical algebra or max-plus algebra. This case is one of our main motivations. As in the case of usual arithmetics, this semifield naturally extends to matrices and vectors, giving rise to **max-linear algebra** [2], and to the tropical convex geometry [4] of max-algebraic subspaces of the nonnegative orthant $\mathbf{R}_+^n$. These subspaces are subsets of $\mathbf{R}_+^n$ closed under taking componentwise maximum, and multiplication by a nonnegative scalar. They are also known as idempotent linear spaces over $\mathbf{R}_+(\max)$[8, 9], or max cones [3].

We give a general proof that in all these cases, any semigroup of commuting matrices has a common eigenvector. This can be seen as an extension of a recent result of Katz, Schneider, Sergeev [6]. We also extend this result on existence of a common eigenvector to the case of nilpotent and quasinilpotent semigroup, in full analogy with nilpotent groups. The notion of quasinilpotent semigroup was suggested by G.B. Shpiz. Our proof technique, in its main part in Section 2, does not use that the spaces are finite-dimensional. However, counterexamples to Lemma 1.1 and Lemma 1.2 in the case of infinite-dimensional Banach spaces are well-known. See also Merlet [10] and Izhakian, Johnson, Kambites [5] for other recent results on existence of common eigenvector of a tropical matrix semigroup.

In the sequel, the cases described above are considered simultaneously. That is, the proofs can be verified for 1) linear spaces over complex field, or 2) for pointed cones in the nonnegative orthant (viewed as "linear spaces" over $\mathbf{R}_+$), or 3) for max cones in the nonnegative orthant (linear spaces over $\mathbf{R}_+(\max)$).

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By **spaces** we mean subspaces/cones in any of the cases written above. A space will be called **nontrivial** if it contains a nonzero vector.

We remark that the theory of more general tropical spaces was developed in [8, 9, 11].

Lemma 1.1. *Let $\{W_n\}$ be a sequence of closed nontrivial spaces such that $W_1 \supseteq W_2 \supseteq W_3 \supseteq \ldots$. Then the intersection $\cap_{i=1}^{\infty} W_i$ is a closed nontrivial space.*

Lemma 1.2. *Let W be a closed space invariant under the action of a linear operator A. Then W contains an eigenvector of A.*

Proof. In the case when A is a complex-valued matrix and W is a subspace in $\mathcal{C}^n$, this result is standard. In the case when A is a nonnegative matrix and W is an invariant cone, see the classical work of Kreĭn, Rutman [7], and for instance, Barker-Schneider [1] for related results. For tropical spaces, more general case was treated by Shpiz [11]. When W is a max cone invariant under max-algebraic multiplication by a nonnegative matrix, this follows from the above result of [11]. □

We conclude with the following lemma, which is evident in all algebraic structures that we consider.

Lemma 1.3. *Let W be an eigenspace of a continuous linear operator A, associated with some eigenvalue. Then*

1. *W is closed;*
2. *Let B be another linear operator such that $AB = BA$. Then W is invariant under B, i.e., $BW \subseteq W$.*

2. Main results

We say that a semigroup $\mathcal{S}$ of linear operators acting in space V has an eigenvector v, if for each $A \in \mathcal{S}$ there is a value $\lambda_A \in \mathbf{R}_+$ such that $Av = \lambda_A v$, and $v \in V$ is nonzero.

Theorem 2.1. *Let $\mathcal{S}$ be a semigroup of commuting linear operators acting in V. Then $\mathcal{S}$ has an eigenvector $v \in V$.*

Proof. Let us consider the collection of all closed nontrivial subspaces of V invariant under the operators of $\mathcal{S}$. Such subspaces will be called invariant by abuse of terminology. This collection contains V, so it is non-empty. In all models described above, the intersection of any chain of closed invariant spaces is a closed invariant space, and it is nontrivial by Lemma 1.1. Then we can apply Zorn's Lemma to get a nontrivial minimal closed invariant space W. Consider a subspace of W consisting of all eigenvectors of a matrix $A \in \mathcal{S}$ associated with an eigenvalue λ_A. Denote this subspace by $V(A, \lambda_A)$. By Lemma 1.3, this is a closed invariant subspace of W, and the minimality implies $W = V(A, \lambda_A)$. As we took an arbitrary $A \in \mathcal{S}$, it follows that $W = V(A, \lambda_A)$ for all $A \in \mathcal{S}$ (and some choice of λ_A). Space W contains a nontrivial vector v, which is a common eigenvector of all matrices in $\mathcal{S}$. □

For a semigroup $\mathcal{S}$, define its subsemigroup $\mathcal{S}^{(k)}$ consisting of all $A \in \mathcal{S}$ that can be represented as $B_1 \cdot \ldots \cdot B_k$ with $B_1, \ldots, B_k \in \mathcal{S}$. Indeed, this is a subsemigroup since $B_1 \cdot \ldots \cdot B_k C_1 \cdot \ldots \cdot C_k$ can be represented, for instance, as $(B_1 \cdot \ldots \cdot B_k C_1) \cdot \ldots \cdot C_k$. $\mathcal{S}$ is called *quasinilpotent* if $\mathcal{S}^{(k)}$ is a commutative semigroup for some k. Recall that a semigroup $\mathcal{S}$ with a zero element 0 is *nilpotent* if $\mathcal{S}^{(k)} = 0$ for some k. Of course, every commutative or nilpotent semigroup is quasinilpotent.

Theorem 2.2. *Let $\mathcal{S}$ be a quasinilpotent (or nilpotent) semigroup of linear operators acting in V. Then $\mathcal{S}$ has an eigenvector $v \in V$.*

Proof. It suffices to prove that if $\mathcal{S}^{(2)}$ has an eigenvector then so does $\mathcal{S}$, and then a straightforward induction can be applied.

Let u be an eigenvector of $\mathcal{S}^{(2)}$ and suppose that there exists $A \in \mathcal{S}^{(2)}$, for which this eigenvector is associated with a nonzero eigenvalue λ. For each $B \in \mathcal{S}$, we have $BA \in \mathcal{S}^{(2)}$ and hence $BAu = \mu u$ for some μ. Substituting $Au = \lambda u$ we obtain $Bu = \lambda^{-1}\mu u$, so u is also an eigenvector of $\mathcal{S}$.

Otherwise, suppose that there is $u \in V$ such that Au is zero for all $A \in \mathcal{S}^{(2)}$. If we also have Bu zero for all $B \in \mathcal{S}$ then u is an eigenvector of $\mathcal{S}$ (with a common eigenvalue equal to zero). Otherwise, take $B' \in \mathcal{S}$ such that $B'u$ is nonzero, but $AB'u$ is still zero for all $A \in \mathcal{S}$. So $B'u$ is an eigenvector of $\mathcal{S}$ (with a common eigenvalue equal to zero). We proved the claim in all possible cases. □

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